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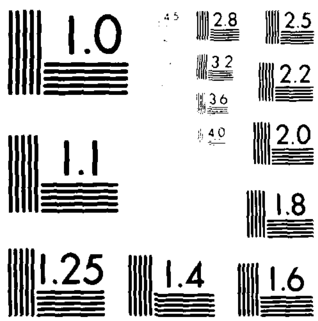

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EXTREMUM PROBLEMS
FOR THE MOTIONS OF A BILLIARD BALL.
IV. A HIGHER-DIMENSIONAL ANALOGUE
OF KEPLER'S STELLA OCTANGULA

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May 1980

(Received March 3, 1980)

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U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
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EXTREMUM PROBLEMS FOR THE MOTIONS OF A BILLIARD BALL. IV. A HIGHER-DIMENSIONAL ANALOGUE OF KEPLER'S STELLA OCTANGULA

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ABSTRACT

Kepler's Stella Octangula (shortened to SO) is the union of the surfaces of the two regular tetrahedra $T = ABCD$ and $T' = A'B'C'D'$ inscribed in the cube

$$\gamma_3 : -1 \leqq x_i \leqq 1, \ (i = 1,2,3)$$

of Figure 1. König and Szücs in [3] have observed that T is obtained by reflexions of the plane of the triangle ABC within γ_3, if we think of the six facets of γ_3 as mirrors which reflect all incident planes back in γ_3. A polyhedron obtained by reflexions of a plane in the facets of γ_3 is called a $K-S$ polyhedron. Thus T, and also T', are $K-S$ polyhedra, and $SO = T \cup T'$.

Let L_3' be a plane intersecting γ_3, which is in general position (G.P.), by which we mean that L_3' is not parallel to any of the three coordinate axes. Let Π_3' denote the $K-S$ polyhedron obtained by reflexions of L_3' in the facets of γ_3, so that $\Pi_3' \subset \gamma_3$. We also say that Π_3' is in G.P. if L_3' is in G.P. We observe that SO does not penetrate within the open cube

$$C_3 : \|x\|_\infty < \frac{1}{3} ,$$

while the 8 vertices of C_3 are all in SO. It is shown that every $K-S$ polyhedron Π_3 in G.P. and having facets different from the 8 facets of SO, must penetrate into the cube C_3.

Using a result from the previous paper [4] we construct an analogue of the SO in $\mathbb{R}^n$, and denote it by SO_n. This analogue is characterized by a property similar to the above, but with respect to the cube

$$C_n : \|x\|_\infty < \frac{1}{n} \text{ in } \mathbb{R}^n .$$

AMS (MOS) Subject Classifications: 51M20, 52A40

Key Words: Extremum problems, Billiard ball motions

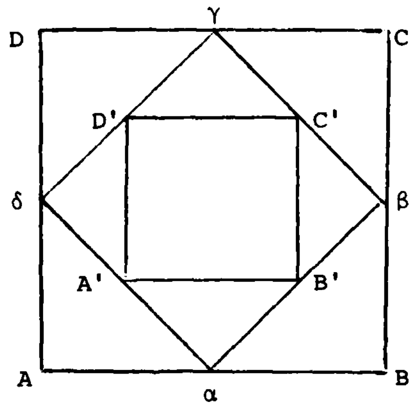
Work Unit Number 3 - Numerical Analysis and Computer Science

Sponsored by the United States Army under Contract Nos. DAAG29-75-C-0024 and DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Let us consider a square billiard table γ_2 = ABCD, and let A'B'C'D' be a concentric square half the size of ABCD. Let α, β, γ, δ, be the midpoints of the sides of γ_2. We observe that the path

gamma



of a billiard ball moving along the sides of the square $\alpha\beta\gamma\delta$ does not penetrate inside the square A'B'C'D'. However, it can be shown that the path of any other billiard ball must penetrate into the square A'B'C'D'. By "any other" we mean

1. That the path is not parallel to any side of γ_2.
2. That the path is different from $\alpha\beta\gamma\delta$.

In the present paper this curious property of plane billiard ball motions is extended to a certain class of skew polytopes in the n-dimensional space $\mathbb{R}^n$. These polytopes reduce to plane billiard ball motions if $n = 2$. If $n = 3$ the above property of the square $\alpha\beta\gamma\delta$ is taken over by Kepler's Stella Octangula. This is an 8-pointed star shown in Figure 1 of the paper, and explains its subtitle.

which is explained in the paper.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXTREMUM PROBLEMS FOR THE MOTIONS OF A BILLIARD BALL

IV. A HIGHER-DIMENSIONAL ANALOGUE OF KEPLER'S STELLA OCTANGULA

I. J. Schoenberg

1. Introduction. The following pages describe what would be merely an exercise in Descriptive Geometry, if it were not for the fact that we are in the space $\mathbb{R}^n$, and are thereby forced to use analytic geometry. The 8-pointed star called Stella Octangula (abbreviated to SO) mentioned in the title, is shown in Figure 1.

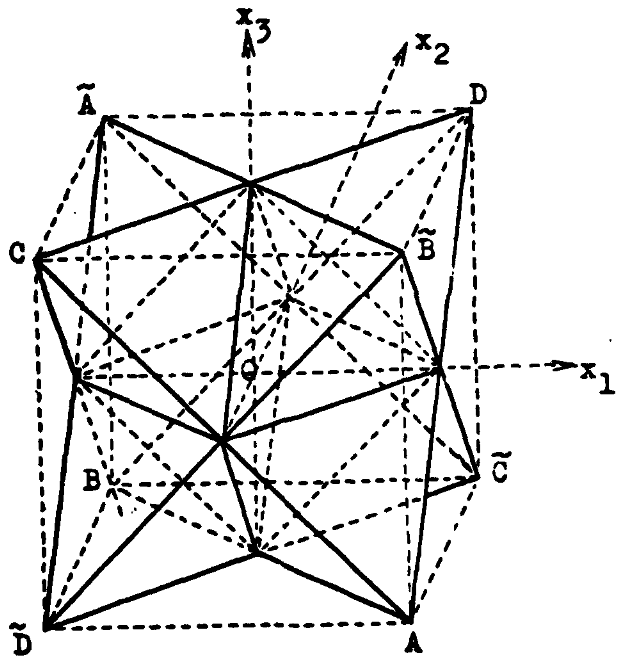


Fig.1

It is composed of the union of the surfaces of the two regular tetrahedra

(1.1) $$\Pi_3 = ABCD, \quad \text{and} \quad \tilde{\Pi}_3 = \tilde{A}\tilde{B}\tilde{C}\tilde{D} \quad ,$$

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both inscribed in the cube γ_3. Notice that Π_3 and $\tilde{\Pi}_3$ are regular simplices of $\mathbb{R}^3$, and that they are symmetric to each other with respect to the center O of γ_3.

My colleagues Carl de Boor and Donald Crowe observed that Kepler's SO is an analogue in $\mathbb{R}^3$ of the star of David, the tetrahedra Π_3, $\tilde{\Pi}_3$, playing the role of the two regular triangles of David's star. An analogue of SO in $\mathbb{R}^n$ now seems obvious: In $\mathbb{R}^n$ we consider two congruent regular simplices α_n and $\tilde{\alpha}_n$ having a common center O and so placed, that they are symmetric to each other with respect to O. This, however, is not the analogue of SO in $\mathbb{R}^n$, that we have in mind.

We denote by SO_n the analogue in $\mathbb{R}^n$ of SO to be now defined, writing $SO_3 = SO$ if $n = 3$. Let

$$\gamma_n = \{-1 \leq x_i \leq 1,\ i = 1,\dots,n\} \tag{1.2}$$

be our fundamental measure polytope, or hypercube. For the natural number j we consider the cross-polytope

$$O_n^j = \{(x_i);\ \sum_1^n |x_i| = 2j - 1\} \tag{1.3}$$

for values of j such that

$$2j - 1 < n \ . \tag{1.4}$$

Its 2^n facets are in the hyperplanes (we abbreviate "hyperplane" to HP, plural HPs).

$$\sum_1^n \epsilon_i x_i = 2j - 1, \quad \text{where } \epsilon_i = \pm 1 \ . \tag{1.5}$$

Its intersection with γ_n we denote by

$$F_j(\epsilon_1,\dots,\epsilon_n) = \gamma_n \cap \{\sum_1^n \epsilon_i x_i = 2j - 1\} \ . \tag{1.6}$$

Notice that this intersection is a non-degenerate convex $(n - 1)$- dimensional polytope. The reason for this is that the vertex $(\epsilon_1,\ldots,\epsilon_n)$ of γ_n, and its center $O = (0,\ldots,0)$ are on opposite sides of the HP (1.5), because we assume that $n > 2j - 1$.

The $F_j(\epsilon_1,\ldots,\epsilon_n)$ are by definition the facets of SO_n; to define SO_n we merely have to form their union

$$SO_n = \bigcup_{2j-1<n} \bigcup_{\epsilon_i=\pm 1} F_j(\epsilon_1,\ldots,\epsilon_n) \quad . \tag{1.7}$$

This, then, is our analogue of SO in $\mathbb{R}^n$.

Let us look at the simplest examples.

1. $n = 2$. The inequality $2j - 1 < n$ is satisfied by the single value $j = 1$. By (1.6) we obtain the edge

$$F_1(\epsilon_1,\epsilon_2) = \gamma_2 \cap \{\epsilon_1 x_1 + \epsilon_2 x_2 = 1\} \quad ,$$

and so (1.7) reduces to

$$SO_2 = \bigcup_{\epsilon_i=\pm 1} F_1(\epsilon_1,\epsilon_2) = \Pi_2 \quad , \tag{1.8}$$

which is the slanting square of Figure 2.

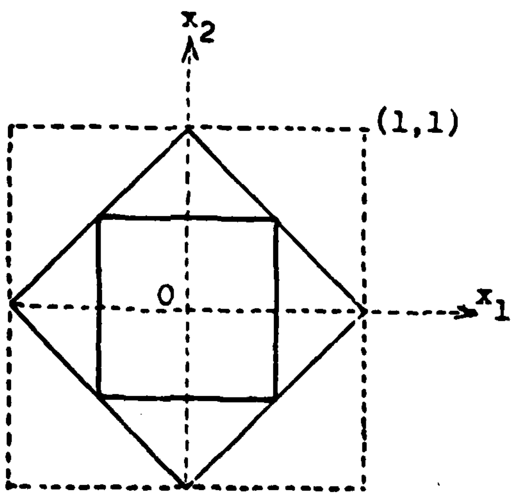

Fig.2

2. $n = 3$. Again $2j - 1 < n$ has the only solution $j = 1$. We see that (1.3), (1.4), reduces to the single octahedron

(1.9) $$O_3^1 = \{(x_1,x_2,x_3);\ \sum_1^3 |x_i| = 1\} \quad ,$$

and this is the regular octahedron whose 6 vertices are the centers of the 6 facets of the cube of Figure 1. According to (1.6) and (1.7) we are to take the 8 HPs of the facets of O_3^1 and form the union of their intersections with γ_3. In this way we get the 8 facets of

(1.10) $$SO_3 = \bigcup_{\epsilon_i=\pm 1} F_1(\epsilon_1,\epsilon_2,\epsilon_3) = \Pi_3 \cup \tilde{\Pi}_3 \quad .$$

The definition (1.7) is seen to lead to Kepler's SO if $n = 3$.

The two cases of $n = 2$ and $n = 3$ are typical of the general situation. The following results will be established.

1. If $n = 2k$ is even, then

(1.11) $$SO_n = \Pi_n$$

is a connected skew polytope in $\mathbb{R}^n$ having $n2^{n-1}$ facets.

2. If $n = 2k + 1$ is odd, then

(1.12) $$SO_n = \Pi_n \cup \tilde{\Pi}_n \quad .$$

Here Π_n and $\tilde{\Pi}_n$ are connected skew polytopes in $\mathbb{R}^n$, which are symmetric to each other with respect to the center O, hence

(1.13) $$\tilde{\Pi}_n = -\Pi_n \quad .$$

Π_n is composed of $(n - 1)2^{n-2}$ facets.

The way the facets $F_j(\epsilon_1,\dots,\epsilon_n)$ of SO_n are distributed among Π_n and $\tilde{\Pi}_n$, is described by the representations

(1.14) $$\Pi_n = \bigcup_{j=1}^{k} \bigcup_{\prod_1^n \epsilon_i = (-1)^j} F_j(\epsilon_1,\dots,\epsilon_n) \quad ,$$

and

(1.15) $$\tilde{\Pi}_n = \bigcup_{j=1}^{k} \bigcup_{\prod_1^n \epsilon_i = (-1)^{j+1}} F_j(\epsilon_1,\dots,\epsilon_n) \quad .$$

The case when $n = 3$, hence $k = 1$, already shows clearly this structure: We divide the 8 facets of the octahedron (1.9) into two classes depending on the sign of the product $\epsilon_1\epsilon_2\epsilon_3$, to obtain

(1.16) $$\Pi_3 = \bigcup_{\epsilon_1\epsilon_2\epsilon_3=-1} F_1(\epsilon_1,\epsilon_2,\epsilon_3)$$

and

(1.17) $$\tilde{\Pi}_3 = \bigcup_{\epsilon_1\epsilon_2\epsilon_3=1} F_1(\epsilon_1,\epsilon_2,\epsilon_3) \quad .$$

The 4 facets of O_3^1 with $\epsilon_1\epsilon_2\epsilon_3 = -1$, and the 4 facets with $\epsilon_1\epsilon_2\epsilon_3 = 1$, form a kind of "checkerboard" design on the surface of O_3^1.

We may assume that $\mathbb{R}^n = \mathbb{R}^{n+1} \cap \{x_{n+1} = 0\}$, and it then follows that $\gamma_n \subset \gamma_{n+1}$. From (1.6) we conclude that

$$F_j(\epsilon_1,\dots,\epsilon_n) \subset F_j(\epsilon_1,\dots,\epsilon_n, \epsilon_{n+1}) \quad ,$$

and finally the definition (1.7) of SO_n proves the inclusion

(1.18) $$SO_n \subset SO_{n+1} \quad .$$

For $n = 2$ the inclusion $SO_2 \subset SO_3$, hence $\Pi_2 \subset \Pi_3 \cup \tilde{\Pi}_3$, is nicely exhibited in Figure 1, where the square Π_2, of Figure 2, appears as the intersection of $SO_3 = \Pi_3 \cup \tilde{\Pi}_3$ with the plane $x_3 = 0$.

We come now to the characteristic properties of Π_n. The polytope Π_n was derived in our previous paper [4, equation (1.15)], and was there denoted by $\tilde{\Pi}_n^{n+1}$. It was there shown that Π_n is a so-called <u>König-Szücs polytope</u>, and that among all such polytopes <u>in general position</u>, it is the one that <u>stays farthest away from the origin</u>. These concepts and results will be discussed in §2. However, our discussion will not be independent of the paper [4], because we take over from [4] the parametric representation of Π_n stated in Theorem 1 below.

The contents of the chapters, sections, and appendix are deemed to be sufficiently explained by their headings.

I am grateful to Professor H. S. M. Coxeter for suggesting the present study of the geometric structure of SO_n based on the parametric representation of Π_n as given in [4]. An altogether different approach to SO_n was given by Coxeter in his paper [1].

I. <u>The skew polytope</u> Π_n

2. <u>A characterization of the skew polytope</u> Π_n <u>of</u> $\mathbb{R}^n$. Let

$$\gamma_n : -1 \leqq x_\nu \leqq 1, \ (\nu = 1,\ldots,n) \tag{2.1}$$

denote the cube I^n, where $I = [-1,1]$. In $\mathbb{R}^n$ we consider the hyperplane (abbreviated to HP, plural HPs) in parametric form

$$L_n' : x_\nu = \sum_{i=1}^{n-1} \lambda_\nu^i u_i + a_\nu, \ (\nu = 1,\ldots,n) \quad , \tag{2.2}$$

where $a = (a_\nu)$ is an interior point of γ_n. It helps to think of L'_n as carrying an $(n-1)$-dimensional pencil of light-rays emanating from the point $a = (a_\nu)$ and spreading uniformly through L'_n. We think of the $2n$ facets $x_\nu = \pm 1$ of γ_n as mirrors which reflect back into γ_n any rays that may strike them, as well as any reflected rays. The complete path of these reflected rays is a skew $(n-1)$-dimensional polytope Π'_n such that

$$\Pi'_n \subset \gamma_n \quad . \tag{2.3}$$

For the two special cases when $n = 2$ and $n = 3$, the skew polygons Π'_2 and skew polyhedra Π'_3 were first considered by D. König and A. Szücs in their pioneering paper [3]. For this reason <u>we refer to</u> Π'_n <u>as a</u> K - S <u>polytope</u>. Observe that Π'_2 is also the path of a billiard ball moving within the square "table" γ_2.

Our first task is to find a convenient representation for the polytope Π'_n. This is nicely obtained by using the <u>reflecting function</u> $R(u)$ defined as follows:

$$R(u) = \begin{cases} u & \text{if } -1 \leqq u \leqq 1, \\ 2-u & \text{if } \quad 1 \leqq u \leqq 3, \end{cases} \quad \underline{\text{and}} \quad R(u+4) = R(u) \quad \underline{\text{for all}} \quad u \quad . \tag{2.4}$$

This function is an appropriate normalization of a so-called <u>linear Euler spline</u>; its graph is shown in Figure 3. Using $R(u)$ it is

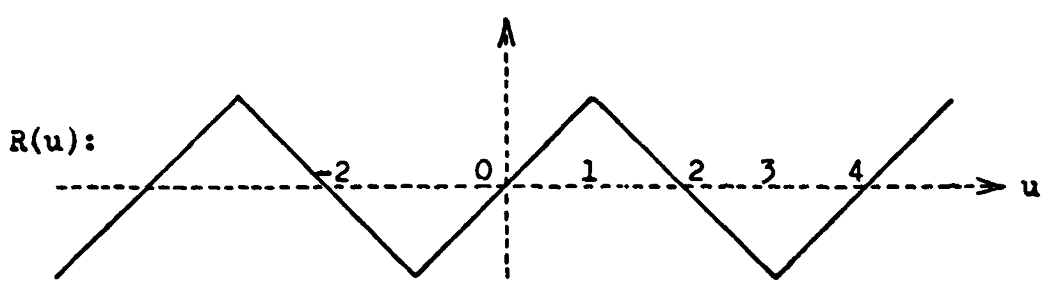

Fig.3

found that a parametric representation of the K - S polytope Π_n' obtained by reflecting the HP (2.2) is given by the equations

$$\Pi_n' : x_\nu = R\left(\sum_{i=1}^{n-1} \lambda_\nu^i u_i + a_\nu\right), \quad (\nu = 1,\ldots,n) \quad . \tag{2.5}$$

The reasons are briefly as follows: By (2.4) we have $R(u) = u$ in the interval $-1 \leqq u \leqq 1$, and this implies, by (2.2), that the intersection $L_n' \cap \gamma_n$ is left pointwise unchanged in passing from (2.2) to (2.5). The remaining portion of Π_n' is obtained by successive reflections of $L_n' \cap \gamma_n$ in the facets of γ_n, due to the zig-zag nature of the graph of Figure 2.

In order to avoid essentially lower-dimensional problems, we assume that the HP (2.2) is not parallel to any of the coordinate axes. The conditions for this are that

(2.6) <u>The</u> $n \times (n-1)$ <u>matrix</u> $\|\lambda_\nu^i\|$ <u>has no vanishing minor of order</u> $n - 1$,

and we then say that L_n', as well as Π_n', are in <u>general position</u>.

Our problem is as follows.

<u>Problem</u> 1. <u>Among all</u> K - S <u>polyhedra</u> Π_n', <u>defined by</u> (2.5), <u>which are in general position, to find those which stay away "as far as possible" from the center</u> O <u>of</u> γ_n.

What does "as far as possible" mean? We use here the Minkowskian norm

$$\|x\|_\infty = \max(|x_1|,\ldots,|x_n|)$$

and determine the open neighborhood $\|x\|_\infty < \rho$, with maximal ρ, which contains no point of Π_n'. We find that

$$\max \rho = \frac{1}{n} \quad , \tag{2.7}$$

and determine the corresponding extremizing Π_n', which we denote by Π_n. This extremum problem, resulting in Theorem 1 below, was solved in our previous paper [4]. In describing its solution, it is convenient to use a different linear Euler spline denoted by $\langle u \rangle$, and related to the function (2.4) by

$$\langle u \rangle = R(2u - 1) \quad .$$

This function may also be defined by

$$(2.8) \qquad \langle u \rangle = \begin{cases} 2u-1 & \text{if} \quad 0 \leqq u \leqq 1, \\ -2u-1 & \text{if} \quad -1 \leqq u \leqq 0, \end{cases} \quad \underline{\text{and}} \quad \langle u+2 \rangle = \langle u \rangle \quad \underline{\text{for all}} \quad u \quad .$$

Its graph is shown in Figure 4

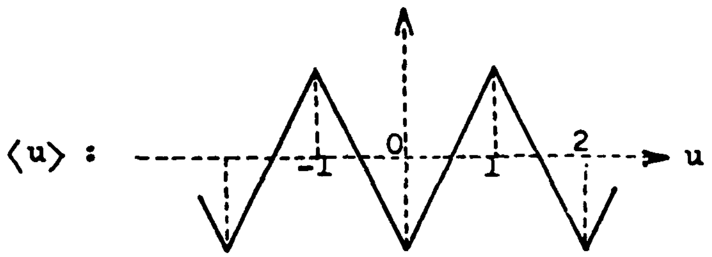

Fig.4

It should be clear that also the equations

$$(2.9) \qquad x_\nu = \left\langle \sum_{i=1}^{n-1} \lambda_\nu^i u_i + a_\nu \right\rangle , \quad (\nu = 1,\dots,n) \quad ,$$

always define a K - S polytope, except that it no longer arises by reflexions of (2.2), but by reflexions of a HP simply related to (2.2). In terms of the function (2.8), the solution of Problem 1 as given in [4, §6], is described by the following

Theorem 1. 1. *The special* K - S *polytope*(*)

$$(2.10)\qquad \Pi_n : \begin{cases} x_i = \langle u_i \rangle \quad (i = 1,\dots,n-1) \\ x_n = \left\langle \sum_1^{n-1} u_i + \frac{n-1}{2} \right\rangle, \quad (0 \leqq u_i \leqq 2, \quad i = 1,\dots,n) \end{cases}$$

is a finite skew polytope which has no point in common with the open cube

$$(2.11)\qquad C_n : \|x\|_\infty < \frac{1}{n} ,$$

so that

$$(2.12)\qquad \Pi_n \cap C_n = \emptyset ,$$

while all 2^n *vertices* $(\pm\frac{1}{n},\dots,\pm\frac{1}{n})$ *of* C_n *are points of* Π_n.

2. *If* (2.2) *is a* HP L_n' *in general position which is different from the HPs of the facets of* Π_n, *and also different from the HPs of the facets of*

$$(2.13)\qquad \tilde{\Pi}_n = -\Pi_n ,$$

then the K - S *polytope* Π_n', *obtained by reflecting* L_n', *must satisfy*

$$(2.14)\qquad \Pi_n' \cap C_n \neq \emptyset ,$$

which means that Π_n' *penetrates into the cube* (2.11).

Remark. Let p satisfy $1 \leqq p \leqq \infty$, and let

$$(2.15)\qquad \|x\|_p = \left(\sum_1^n x_\nu^p\right)^{1/p} = \left(\sum_1^n \left(\frac{1}{n}\right)^p\right)^{1/p} = \frac{1}{n^{1-\frac{1}{p}}}$$

(*) In [4, equations (1.3)] we used a different normalization of our present function (2.8); let us denote the old function for the moment by $\langle\!\langle u \rangle\!\rangle$. In terms of the present function $\langle u \rangle$ its expression is $\langle\!\langle u \rangle\!\rangle = \frac{1}{2}(\langle u \rangle + 1)$. This reflecting function was adapted to the measure-polytope $\gamma_n' : \{0 \leqq x_\nu \leqq 1,\ \nu = 1,\dots,n\}$, which we now abandon in favor of (2.1). It should not be surprizing that our equations (2.10) are identical with the old equations (1.15) of [4].

denote the p-sphere circumscribed to the cube (2.11).

In §11 we show that Theorem 1 remains correct if we replace in its statement the cube C_n by the open p-neighborhood

(2.16) $$\|x\|_p < \frac{1}{n^{1-\frac{1}{p}}}, \quad (1 \leq p \leq \infty) \quad .$$

II. Symmetry properties of the polytope Π_n

3. Π_n is invariant on permutations of the axes. We state this property as

Lemma 1. If

(3.1) $$(x_1, \ldots, x_n) \in \Pi_n \quad ,$$

and $(x_{i_1}, \ldots, x_{i_n})$ is a permutation of the coordinates $(x_1, \ldots, x_n)$, then

(3.2) $$(x_{i_1}, \ldots, x_{i_n}) \in \Pi_n \quad .$$

Proof. This will become evident as soon as we rewrite the equations (2.10) in a symmetric form. We introduce the new parameter u_n by the equation

(3.3) $$\sum_1^n u_\nu = -\frac{n-1}{2} \quad ,$$

and (2.10) show that

$$x_n = \left\langle \sum_1^{n-1} u_i + \frac{n-1}{2} \right\rangle = \langle -u_n \rangle = \langle u_n \rangle \quad .$$

Therefore the equations (2.10) may be replaced by the symmetric system

(3.4) $$x_\nu = \langle u_\nu \rangle, \quad (\nu = 1, \ldots, n) \quad ,$$

where the n parameters u_ν are connected by the equation (3.3). Since (3.4) is symmetric in the u, the lemma has become evident.

4. The symmetries of the polytope Π_n. These depend strongly on the parity of n.

Lemma 2. We assume that

(4.1) $$n = 2k \text{ is even.}$$

The polytope Π_n remains invariant if we perform a reflexion in the hyperplane $x_\nu = 0$.

Proof. By Lemma 1 we may assume that $\nu = 1$, and we are to show that

(4.2) $$(x_1,\ldots,x_n) \in \Pi_n \text{ implies that } (-x_1,x_2,\ldots,x_n) \in \Pi_n \; .$$

By (2.10) and (4.1) let

$$x_i = \langle u_i \rangle, \; x_n = \langle \sum_1^{n-1} u_i + k - \frac{1}{2} \rangle$$

and

$$x_i' = \langle u_i' \rangle, \; x_n' = \langle \sum_1^{n-1} u_i' + k - \frac{1}{2} \rangle$$

be two points of Π_n, where

$$u_1' = 1-u_1, \; u_i' = -u_i \; (i = 2,\ldots,n-1) \; .$$

From Figure 4 $\langle u \rangle$ is seen to be odd about the point $u = \frac{1}{2}$, and so

$$x_1' = \langle u_1' \rangle = \langle 1-u_1 \rangle = -\langle u_1 \rangle = -x_1 \; ,$$

while

$$x_i' = \langle u_i' \rangle = \langle -u_i \rangle = \langle u_i \rangle = x_i \text{ for } i = 2,\ldots,n-1 \; .$$

Finally

$$x_n' = \langle \sum_1^{n-1} u_i' + k - \frac{1}{2}\rangle = \langle 1 - \sum_1^{n-1} u_i + k - \frac{1}{2}\rangle = \langle k + \frac{1}{2} - \sum_1^{n-1} u_i \rangle$$

$$\langle \sum_1^{n-1} u_i - k - \frac{1}{2}\rangle = \langle \sum_1^{n-1} u_i + k - \frac{1}{2}\rangle = x_n \quad ,$$

proving (4.2).

A consequence of Lemma 2 is

Corollary 1. *If* n *is even, then* Π_n *has the origin* O *as center of symmetry, hence*

$$\Pi_n = -\Pi_n \quad . \tag{4.3}$$

For odd n we have

Lemma 3. *Let*

$$n = 2k+1 \quad \textit{be odd}. \tag{4.4}$$

The polytope Π_n *remains invariant if we perform a reflexion in the hyperplane* $x_i = 0$ *followed by a reflexion in* $x_j = 0$ $(j \neq i)$.

Proof. Again in view of Lemma 1 we may assume that $i = 1$ and $j = 2$, and we are to show that the mapping

$$(x_1,x_2,\ldots,x_n) \to (-x_1,-x_2,x_3,\ldots,x_n) \quad \textit{leaves } \Pi_n \textit{ invariant}. \tag{4.5}$$

By (4.4) the equations (2.10) become

$$x_i = \langle u_i \rangle, \; x_n = \langle \sum_1^{n-1} u_i + k \rangle \quad .$$

However, the identity $\langle u+1 \rangle = -\langle u \rangle$ shows that $\langle u+k \rangle = (-1)^k \langle u \rangle$, and so we may replace (2.10) by

$$(4.6)\qquad \Pi_n : \quad \begin{aligned} x_i &= \langle u_i \rangle, \quad (i = 1,\ldots,n-1) \quad , \\ x_n &= (-1)^k \langle \sum_1^{n-1} u_i \rangle \quad . \end{aligned}$$

Besides the point (x_ν) defined by (4.6), we consider a second point (x'_ν) corresponding to parameters u'_i defined by

$$u'_1 = u_1+1,\ u'_2 = u_2-1,\ u'_i = u_i \quad (i = 3,\ldots,n-1) \quad .$$

From the equations

$$x'_1 = \langle u'_1 \rangle = \langle u_1+1 \rangle = -\langle u_1 \rangle = -x_1 \quad ,$$

$$x'_2 = \langle u'_2 \rangle = \langle u_2-1 \rangle = -\langle u_2 \rangle = -x_2 \quad ,$$

$$x'_i = x_i \quad (i = 3,\ldots,n-1) \quad ,$$

$$x'_n = (-1)^k \langle \sum_1^{n-1} u'_i \rangle = (-1)^k \langle \sum_1^{n-1} u_i \rangle = x_n \quad ,$$

we see that (4.5) indeed holds.

We have the

Corollary 2. *Let (4.4) hold, and let us define the symmetric image of Π_n with respect to the origin by*

$$(4.7)\qquad \tilde{\Pi}_n = -\Pi \quad .$$

Then a reflexion of Π_n in a coordinate hyperplane carries Π_n into $\tilde{\Pi}_n$.

Proof. By Lemma 3, and because n is odd, we have

$$(4.8)\qquad (x_1,x_2,\ldots,x_n) \in \Pi_n \quad .$$

if and only if $(x_1,-x_2,\ldots,-x_n) \in \Pi_n$. Now (4.7) shows that the last inclusion holds iff

$$(4.9)\qquad (-x_1,x_2,\ldots,x_n) \in \tilde{\Pi}_n \quad .$$

The equivalence of (4.8) with (4.9) proves our corollary.

III. The geometric structure of Π_n revealed

We separate the discussion according to the parity of n.

5. The case when $n = 2k$ is even. From $n = 2k$ and properties of $\langle u\rangle$, the equations (2.10) defining Π_n may be written

$$(5.1) \qquad \Pi_{2k} : \begin{cases} x_i = \langle u_i\rangle, \quad (i = 1,\dots,n-1) , \\ x_n = (-1)^k \langle \frac{1}{2} - \sum_1^{n-1} u_i\rangle . \end{cases}$$

On the other hand we have defined the analogue of Kepler's Stella Octangula by the equation (1.7). As j ranges over the natural numbers satisfying $2j - 1 < n = 2k+1$, we find that $j = 1,2,\dots,k$, and so we may rewrite (1.7) as

$$(5.2) \qquad SO_{2k} = \bigcup_{j=1}^{k} \bigcup_{\epsilon_i=\pm 1} F_j(\epsilon_1,\dots,\epsilon_{2k}) ,$$

where

$$(5.3) \qquad F_j(\epsilon_1,\dots,\epsilon_{2k}) = \gamma_{2k} \cap \{\sum_1^{2k} \epsilon_i x_i = 2j-1\} .$$

Our main result for even n is

Theorem 2. We have

$$(5.4) \qquad \Pi_{2k} = SO_{2k} .$$

Proof. Observe that Π_{2k} as defined by (5.1) and expressed in terms of $\langle u\rangle$, is automatically a K - S polytope. On the other hand SO_{2k}, as defined by (5.2), appears as just a collection of $k2^{2k}$ facets about which, a priori, we have no idea how they hang together, if at all. We will show, however, that the facets of SO_{2k} are precisely the facets of Π_{2k}.

We start by identifying an obvious facet of Π_{2k}, and we do this by restricting the u_i to satisfy the inequalities

(5.5) $$0 \leqq u_i \ (i = 1,\dots,n-1), \ \sum_1^{n-1} u_i \leqq \frac{1}{2} \ .$$

Figure 4 and (5.1) show that the x_ν may be explicitly expressed in terms of the u_i by using $\langle u \rangle = 2u-1$ if $0 \leqq u \leqq 1$. It follows that the image of the simplex (5.5) is in the HP

$$x_i = 2u_i-1, \ (-1)^k x_n = 1 - 2\sum_1^{n-1} u_i - 1 = -2\sum_1^{n-1} u_i \ .$$

Eliminating the u_i we find that $(-1)^k x_n = \sum_1^{n-1} (-1-x_i) = -(n-1) - \sum_1^{n-1} x_i$,

and if follows that the HP

$$-x_1 - x_2 - \dots - x_{n-1} - (-1)^k x_n = n-1$$

contains a facet of Π_{2k}. Applying to it the arbitrary reflexions in the coordinate HPs allowed by Lemma 2, we conclude that the 2^n HPs

(5.6) $$\sum_1^n \epsilon_i x_i = n-1 \ (= 2k-1)$$

contain facets of Π_n. Notice that <u>these are precisely the facets</u>

(5.7) $$\bigcup_{\epsilon_i=\pm 1} F_k(\epsilon_1,\dots,\epsilon_{2k})$$

<u>of the last term</u> (for $j = k$) <u>of the union</u> (5.2).

The question arises:

(5.8) <u>Where do we go from here by reflexions</u>?

An answer depends on which facets of γ_n the facets of (5.7) intersect. And by intersect we mean <u>strictly</u>, i.e. intersecting the <u>interior</u> of the facet

$\gamma_n \cap \{x_\nu = \eta\}$, $(\eta = \pm 1)$. Incidentally, we will just write $x_\nu = \eta$, meaning thereby the facet $\gamma_n \cap \{x_\nu = \eta\}$.

An answer to the question (5.8) is provided by

Lemma 4. The hyperplane (5.6) intersects strictly the facets

$$x_i = \epsilon_i, \ (i = 1,\ldots,n) \quad , \tag{5.9}$$

but (5.6) has no point in common with the remaining n facets of γ_n.

Proof. 1. We may ssume that $i = 1$. The intersection of (5.6) with the facet $x_1 = \epsilon_1$ is in the HP

$$\sum_2^n \epsilon_i x_i = n-2 \quad ,$$

and this equation has solutions in the open cube

$$-1 < x_2 < 1, \ -1 < x_3 < 1, \ldots, \ -1 < x_n < 1 \quad ,$$

for instance the point $x_i = (n-2)(n-1)^{-1}\epsilon_i$ $(i = 2,\ldots,n)$.

2. On the other hand, the intersection of (5.6) with the facet $x_1 = -\epsilon_1$ is in the HP

$$\sum_2^n \epsilon_i x_i = n$$

and this equation has evidently no solutions in the closed cube

$$-1 \leqq x_2 \leqq 1, \ldots, -1 \leqq x_n \leqq 1 \quad ,$$

because its left side has only $n - 1$ terms, all of absolute value ≤ 1.

Let the HP

$$\sum_1^n \epsilon_i x_i = C \tag{5.10}$$

intersect the interior of γ_n so as to produce the facet

$$F = \gamma_n \cap \{\sum \epsilon_i x_i = C\} \quad .$$

What are the reflexions of F in the facet of γ_n? The answer is given by

Lemma 5. 1. If (5.10) intersects strictly the facet $x_1 = \epsilon_1$, then its reflexion in $x_1 = \epsilon_1$ is in the HP

$$(5.11) \qquad -\epsilon_1 x_1 + \sum_2^n \epsilon_i x_i = C - 2 \quad .$$

2. If (5.10) intersects strictly the facet $x_1 = -\epsilon_1$, then its reflexion in $x_1 = -\epsilon_1$ is in the HP

$$(5.12) \qquad -\epsilon_1 x_1 + \sum_2^n \epsilon_i x_i = C + 2 \quad .$$

Proof: 1. To perform the reflexion it is convenient to shift the origin to the point $(\epsilon_1, 0, \ldots, 0)$ by writing (5.10) in the form

$$\epsilon_1(x_1 - \epsilon_1) + \sum_2^n \epsilon_i x_i = C - 1 \quad .$$

We now obtain the equation of the reflected HP by changing the sign of the factor $(x_1 - \epsilon_1)$ to obtain $-\epsilon_1(x_1 - \epsilon_1) + \sum_2^n \epsilon_i x_i = C - 1$, and finally (5.11). 2. Likewise, to reflect (5.10) in $x_1 = -\epsilon_1$, we write (5.10) as $\epsilon_1(x_1 + \epsilon_1) + \sum_2^n \epsilon_i x_i = C + 1$, to obtain the reflected HP $-\epsilon_1(x_1 + \epsilon_1) + \sum_2^n \epsilon_i x_i = C + 1$, and finally (5.12).

By Lemma 4 we can reflect (5.6) in $x_1 = \epsilon_1$; setting $C = n-1 = 2k-1$ we obtain from (5.11), by applying the reflexions of Lemma 2, the entire collection of 2^n HPs

$$(5.13) \qquad \sum_1^n \epsilon_i x_i = 2k-3, \quad \text{for arbitrary } \epsilon_i = \pm 1 \quad .$$

If we reflect this HP in $x_1 = -\epsilon_1$, we return to the HPs (5.6). However, if $n \geq 6$, and if we reflect (5.13) in $x_1 = \epsilon_1$, we obtain, again via Lemma 2, the collection of 2^n HPs

$$\sum_1^n \epsilon_i x_i = 2k-5, \quad \text{for arbitrary } \epsilon_i = \pm 1 \quad . \tag{5.14}$$

We can continue this process until we reach the 2^n HPs

$$\sum_1^n \epsilon_i x_i = 1 \quad . \tag{5.15}$$

We claim that <u>from this point on no further HPs will appear by reflexions</u>. Indeed, reflecting (5.15) in $x_1 = \epsilon_1$, we obtain by Lemma 5, for $C = 1$, the HP

$$-\epsilon_1 x_1 + \sum_2^n \epsilon_i x_i = -1 \quad ,$$

which is already among the HPs (5.15). Reflexion in $x_1 = -\epsilon_1$ will lead to a HP (5.10) with $C = 3$, which was already obtained before.

Our discussion shows that SO_{2k}, of (5.2), is a K - S polytope which is identical with Π_{2k}, proving Theorem 2.

6. <u>The case when</u> $n = 2k+1$ <u>is odd</u>. We found in (4.6) that we can write

$$\Pi_{2k+1}: \quad \begin{aligned} &x_i = \langle u_i \rangle, \ (i = 1,\dots,n-1) \quad , \\ &x_n = (-1)^k \langle \sum_1^{n-1} u_i \rangle \quad . \end{aligned} \tag{6.1}$$

In (1.7) we have defined SO_n, which in our case when $n = 2k+1$, becomes

$$SO_{2k+1} = \bigcup_{j=1}^{k} \bigcup_{\epsilon_i = \pm 1} F_j(\epsilon_1,\dots,\epsilon_{2k+1}) \quad . \tag{6.2}$$

In (1.14) and (1.15) we have decomposed this union into two parts

$$SO_{2k+1} = \sum_0 \cup \sum_1 \quad , \tag{6.3}$$

where

$$\sum_0 = \bigcup_{j=1}^{k} \bigcup_{\Pi\epsilon_i=(-1)^j} F_j(\epsilon_1,\ldots,\epsilon_{2k+1}) \tag{6.4}$$

and

$$\sum_1 = \bigcup_{j=1} \bigcup_{\Pi\epsilon_i=(-1)^{j+1}} F_j(\epsilon_1,\ldots,\epsilon_{2k+1}) \quad . \tag{6.5}$$

Here we wish to prove

<u>Theorem 3</u>. <u>We have</u>

$$\Pi_{2k+1} = \sum_0 \quad \underline{\text{and}} \quad \tilde{\Pi}_{2k+1} = \sum_1 \quad . \tag{6.6}$$

<u>Proof</u>. This is a variation of our proof of Theorem 2. We begin by identifying a certain set of facets of Π_{2k+1}. Restricting the u_i to the simplex

$$0 \leqq u_i, \quad \sum_1^{n-1} u_i \leqq 1 \quad , \tag{6.7}$$

and expressing the x_ν of (6.1), using (2.8), in terms of the u_i, we find on eliminating the u_i between these n equations, that the simplex (6.7) is mapped by (6.1) into the HP

$$-x_1 - x_2 - \cdots - x_{n-1} + (-1)^k x_n = n-2 \ (= 2k-1) \quad . \tag{6.8}$$

Notice that the product of the coefficients of the left side = $(-1)^k$, because $n - 1$ is even. Also, because $n = 2k+1$, we can no longer use Lemma 2, but must appeal to Lemma 3, with the result that from (6.8) we get the collection of 2^{n-1} HPs

$$(6.9) \qquad \sum_1^n \epsilon_i x_i = 2k-1, \quad \underline{\text{where}} \quad \prod_1^n \epsilon_i = (-1)^k \quad .$$

Lemma 5 remains valid. From (5.11) and (5.12), we see that a reflexion in $x_1 = \epsilon_1$, or $x_1 = -\epsilon_1$, will change the sign fixed sign of the product $\prod_1^n \epsilon_i$ for the successive families of HPs (5.13) and (5.14) thus reached.

In this way we find that the collection of facets (6.4) is closed with respect to reflexions. It follows that $\sum_0$ is a finite K - S polytope which must be identical with the K - S polytope Π_{2k+1}. This proves the first identity (6.6). Finally, it should be clear from (6.4) and (6.5) that

$$\sum_1 = -\sum_0 \quad .$$

In view of $\tilde{\Pi}_{2k+1} = -\Pi_{2k+1}$, the second identity (6.6) follows from the first.

7. <u>The polytope</u> Π_n <u>and the cube</u> C_n <u>are disjoint</u>. It seems worthwhile to point out that the results of Part III immediately imply the property (2.12) of Theorem 1, to the effect that Π_n does not penetrate into the open hypercube

$$(7.1) \qquad C_n : \|x\|_\infty < \frac{1}{n} \quad .$$

<u>Let</u> $n = 2k$ <u>be even</u>. Among the facets of SO_n as exhibited in (5.2), the facets $F_1(\epsilon_1,\ldots,\epsilon_n)$ are nearest the origin O. The HP of $F_1(\epsilon_1,\ldots,\epsilon_n)$ has the equation

$$(7.2) \qquad \sum_1^n \epsilon_i x_i = 1 \quad ,$$

and it evidently contains the vertex

$$(7.3) \qquad \left(\frac{\epsilon_1}{n}, \frac{\epsilon_2}{n}, \ldots, \frac{\epsilon_n}{n}\right) \text{ of } C_n \quad .$$

Therefore (7.2) is seen to be the HP through (7.3) and perpendicular to the diagonal of C_n joining its center O to its vertex (7.3).

If $n = 2k+1$ is odd, the situation is similar, in view of (6.4), the only difference being that we consider only such HPs (7.2), and vertices (7.3), that satisfy the condition

$$\epsilon_1\epsilon_2\cdots\epsilon_n = -1 \quad .$$

IV. The true shape and size of the facets of Π_n

8. A choice of coordinates in the parameter space $\mathbb{R}^{n-1}$. As throughout this paper, our foundation is the representation

$$\text{(8.1)} \qquad \Pi_n : \begin{cases} x_i = \langle u_i \rangle, \quad (i = 1,\ldots,n-1) \quad , \\ x_n = \langle \sum_1^{n-1} u_i + \frac{n-1}{2} \rangle \quad , \end{cases}$$

our objective being to describe geometrically the mapping

$$\text{(8.2)} \qquad F : (u_i) \mapsto (x_\nu) \quad .$$

This will be a piecewise isometry, provided that we select in $\mathbb{R}^{n-1}$ a coordinate system as follows.

Let α_{n-1} be a regular simplex in $\mathbb{R}^{n-1}$ such that

$$\text{(8.3)} \qquad \underline{\text{all edges of}}\ \alpha_{n-1}\ \underline{\text{are}} = \sqrt{8} \quad .$$

Let O be one of its vertices and let $f_1, f_2,\ldots,f_{n-1}$ denote the vectors representing its $n - 1$ edges insueing from O. The point $u = (u_i)$ is then represented by

$$\text{(8.4)} \qquad u = \sum_1^{n-1} f_i u_i \quad .$$

From our choice of the f_i we have, in terms of inner products, the equations

$$f_i \cdot f_i = f_i^2 = 8, \quad f_i \cdot f_j = \sqrt{8}\,\sqrt{8}\cos 60^0 = 8 \cdot \frac{1}{2} = 4 \quad . \tag{8.5}$$

The mapping (8.2), explicitly given by (8.1), is piecewise linear due to the presence of the function $\langle u \rangle$. In particular (8.1) is continuous, and has everywhere continuous (in fact constant) partial derivatives $\partial x_\nu / \partial u_i$, with the exception of the hyperplanes on which the expressions inside the function $\langle \cdot \rangle$ assumes integer values. These HPs are

$$u_i = j \qquad (j \in \mathbb{Z},\ i = 1,\ldots,n-1) \tag{8.6}$$

and

$$\sum_1^{n-1} u_i + \frac{n-1}{2} = j, \quad (j \in \mathbb{Z}) \quad . \tag{8.7}$$

If the point $u = (u_i)$ is in none of these HPs we have by (8.1) and (2.8) that

$$dx_i = \pm\, 2\, du_i, \quad dx_n = \pm\, 2 \sum_1^{n-1} du_i$$

and therefore

$$(dx)^2 = \sum_1^n (dx_\nu)^2 = 4 \sum_1^{n-1} (du_i)^2 + 4\Big(\sum_1^{n-1} du_i\Big)^2$$

and finally

$$(dx)^2 = 8 \sum_1^{n-1} (du_i)^2 + 8 \sum_{i<j} du_i du_j \quad . \tag{8.8}$$

On the other hand, from (8.4)

$$(du)^2 = \Big(\sum_1^{n-1} f_i du_i\Big)^2 = \sum_1^{n-1} f_i^2 (du_i)^2 + 2 \sum_{i<j} (f_i \cdot f_j) du_i du_j \quad ,$$

and the equations (8.5) show that

$$(8.9) \qquad (du)^2 = 8\sum_{1}^{n-1} (du_i)^2 + 8\sum_{i<j} du_i du_j \quad .$$

The identity of the quadratic forms (8.8) and (8.9) shows that $|dx| = |du|$, and therefore the mapping (8.2) is an <u>isometry</u> in each of the cells into which the HPs (8.6) and (8.7) divide the space $\mathbb{R}^{n-1}$.

We state our result as

<u>Lemma</u> 6. <u>The cells into which the</u> HPs (8.6) <u>and</u> (8.7) <u>dissect the space</u> $\mathbb{R}^{n-1}$ <u>represent in true shape and size the facets of the skew polytope</u> Π_n.

Let us look more closely at the dissection of $\mathbb{R}^{n-1}$ by the HPs (8.6), (8.7). The HPs of the system (8.6) being parallel to the oblique coordinate HPs, divide $\mathbb{R}^{n-1}$ into a lattice of congruent acute rhombohedra, the fundamental one being

$$(8.10) \qquad Rh_0 = \{(u_i) \; ; \; 0 \leqq u_i \leqq 1, \; i = 1,\ldots,n-1\} \quad .$$

Figure 5 represents in parallel projection Rh_0 for the case $n = 4$. The location of the second system (8.7) depends on the parity of n. Accordingly our discussion branches out into two cases.

9. <u>The dimension</u> $n = 2k$ <u>is even</u>. Now (8.7) becomes

$$(9.1) \qquad \sum_{1}^{n-1} u_i = j + \frac{1}{2}, \; (j \in \mathbb{Z}) \quad .$$

Since the sum $\sum u_i$ varies in the rhombohedron (8.10) from the value zero, at O, to the value $n - 1$ at the opposite vertex $\sum f_i$, we see that the HPs (9.1) that intersect Rh_0 are the $n - 1$ HPs

$$(9.2) \qquad \sum_{1}^{n-1} u_i = j + \frac{1}{2} \quad \text{for} \quad j = 0,1,\ldots,n-2 \quad .$$

These $n - 1$ HPs dissect Rh_0 into n cells. By Lemma 4 these n cells represent in true shape and size n facets of the polytope Π_n.

What are all the facets of Π_n, for $n = 2k$, and how many are there? Rh_0 contains just n of these facets. To obtain them all, we recall that $\langle u \rangle$ has the period 2. We would therefore expect a fundamental region of the mapping (8.1) to be the domain

$$(9.3) \qquad D_{n-1} = \{(u_i);\ 0 \leqq u_i \leqq 2,\ i = 1,\ldots,n-1\} \ .$$

This is again a thombohedron of twice the linear size of Rh_0 and expressible as

$$D_{n-1} = \bigcup_{\eta_i=0,1} (Rh_0 \oplus \sum_{1}^{n-1} f_i \eta_i) \ .$$

Since each of these 2^{n-1} unit rhombohedra contains n cells, we conclude that

(9.4) <u>the total number of facets of</u> Π_n <u>is</u> $= n2^{n-1}$.

This agrees with the number given in Theorem 2 and therefore shows that all these facets are different.

Since the mapping (8.1) has the period 2 in each of the variables u_i, we conclude that opposite facets of D_{n-1} are to be identified, and we obtain the following:

<u>Theorem</u> 4. <u>If</u> $n = 2k$ <u>is even, then the skew polytope</u> Π_n <u>is topologically a torus</u> T^{n-1}.

For $n = 4$ Figure 5 shows that Rh_0 is divided by the 3 planes (9.2) into 4 cells of which the first and fourth are regular tetrahedra having edges $= \sqrt{2}$, while the second and third are truncated tetrahedra, each

bounded by 4 regular triangles and 4 regular hexagons. By (9.4) Π_4 has a total of $4 \times 8 = 32$ facets of which 16 are tetrahedra and 16 are truncated tetrahedra.

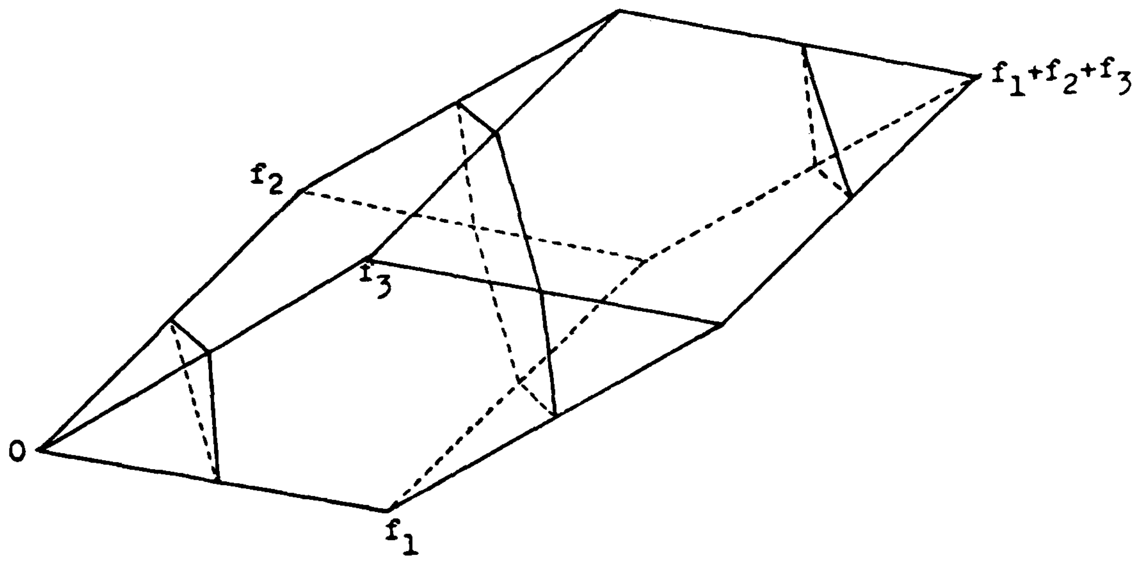

Fig.5

10. <u>What is the topological structure of</u> Π_{2k+1}? In this case of n odd (8.7) becomes $\sum u_i = j$ $(j \in \mathbb{Z})$, and exactly $n - 2$ among them, namely

(10.1) $$\sum_{1}^{n-1} u_i = j \quad (j = 1,2,\ldots,n-2) \quad .$$

dissect the rhombohedron (8.10) into $n - 1$ cells. We would expect a fundamental domain of the mapping (8.1) to be given by (9.3). However, this is not the case due to the following:

<u>Lemma</u> 7. <u>Let</u> $n = 2k+1$ <u>be odd</u>. <u>The mapping</u> (8.1) <u>is even about every lattice point</u> $(u_i) = (j_1,j_2,\ldots,j_{n-1})$, $(j_i \in \mathbb{Z})$.

<u>Proof</u>: We found in (6.1) that (8.1) may be written as

$$
\text{(10.2)} \qquad \Pi_n : \quad \begin{aligned} x_i &= \langle u_i \rangle \\ x_n &= (-1)^k \langle \sum_1^{n-1} u_i \rangle \ . \end{aligned}
$$

The points (u_i) and (u_i') are symmetric in the point (j_i) provided that

$$
u_i' = 2j_i - u_i, \quad (i = 1,\dots,n-1) \quad ,
$$

and we are to show that this implies that

$$
x_\nu' = x_\nu, \quad (\nu = 1,\dots,n) \quad .
$$

This follows from

$$
x_i' = \langle u_i' \rangle = \langle 2j_i - u_i \rangle = \langle -u_i \rangle = \langle u_i \rangle = x_i
$$

if $i < n$, and

$$
x_n' = (-1)^k \langle \sum_1^{n-1} u_i' \rangle = (-1)^k \langle 2 \sum_{i=1}^{n-1} j_i - \sum_1^{n-1} u_i \rangle = (-1)^k \langle \sum_1^{n-1} u_i \rangle = x_n \quad .
$$

In particular, the mapping (10.2) is even about the point $(u_i) = (1,1,\dots,1)$, which is the center of the rhombohedron (9.3). But then we can certainly reduce D_{n-1} to one of its two halves, namely

$$
\text{(10.3)} \qquad D_{n-1}^* = \{(u_i);\ 0 \leqq u_i \leqq 2,\ i = 1,\dots,n-2,\ 0 \leqq u_{n-1} \leqq 1\} \quad ,
$$

and still obtain the complete Π_n as the image of D_{n-1}^*.

In D_{n-1}^* we have the union of 2^{n-2} unit rhombohedra. Because (10.1) and their analogues, dissect each of these into $n - 1$ cells, we get for Π_n a total of $(n-1)2^{n-2}$ facets. This agrees with the number given in Theorem 3 and shows that all these facets are different.

We turn now to the topological structure of Π_n. In the parallelepiped D_{n-1}^* of "height" = 1 we consider $n - 2$ pairs of opposite facets

(10.4) $u_i = 0$ and $u_i = 2$, $(i = 1,\ldots,n-2)$,

and also the <u>top</u> $u_{n-1} = 1$ and the <u>bottom</u> $u_{n-1} = 0$. By the periodicity of (10.2), and by Lemma 7, we are to

1. Identify pairs of opposite facets (10.4) ,

2. Identify two points of the top $u_{n-1} = 1$ that are symmetric in its center $(1,1,\ldots,1,1)$. Likewise identify two points of $u_{n-1} = 0$ that are symmetric in its center $(1,\ldots,1,0)$.

I am unable to identify the topological structure of the fundamental domain D^*_{n-1} with the above identifications of its boundary. Accordingly, we close Part IV with the following unsolved:

<u>Problem</u> 2. <u>To determine the topological structure of the polytope</u> Π_{2k+1}.

V. Appendix

11. <u>Replacing the norm</u> $\|x\|_\infty$ <u>in Theorem 1 by</u> $\|x\|_p$. Here we wish to justify the remark at the end of §2. Let us first circumscribe a p-sphere $\|x\|_p = \rho_p$ to our cube

$$C_n : \|x\|_\infty < \frac{1}{n} . \tag{11.1}$$

The p-norm $(1 \leqq p < \infty)$ of its vertices $(\pm \frac{1}{n}, \ldots, \pm \frac{1}{n})$ is

$$\rho_p = (n \frac{1}{n^p})^{1/p} = 1/n^{1-\frac{1}{p}} , \tag{11.2}$$

and so the open p-sphere circumscribed to C_n is

$$S_p : \|x\|_p < 1/n^{1-\frac{1}{p}} . \tag{11.3}$$

Since S_p is convex we certainly have the inclusion

$$C_n \subset S_p . \tag{11.4}$$

<u>Let us show that</u> *)

$$S_p \subset S_1 = \{\sum_1^n |x_i| < 1\} . \tag{11.5}$$

<u>Proof</u>: This follows from the monotonicity of the ordinary means

$$M_p(|x_i|) = (\frac{1}{n} \sum_1^n |x_i|^p)^{1/p} \quad \text{as functions of } p$$

as shown in [2, §2.9, p. 26]. This monotonicity implies that $M_1(|x_i|) \leqq M_p(|x_i|)$, hence that

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*) This is also evident geometrically from the convexity of S_p and because the 2^n facets of S_1 are the HPs of support of S_p at the 2^n vertices of C_n.

$$\frac{1}{n}\sum_1^n |x_i| \leqq \left(\frac{1}{n}\sum_1^n |x_i|^p\right)^{1/p}$$

or

$$\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p} \geqq \frac{1}{n^{1-\frac{1}{p}}}\sum_1^n |x_i| \quad .$$

But then the inclusion $x = (x_1,\ldots,x_n) \in S_p$, which means that $\|x\|_p < 1/n^{1-\frac{1}{p}}$, surely implies that $\sum_1^n |x_i| < 1$ and (11.5) is established.

<u>A third preliminary remark is that</u>

(11.6) $$\Pi_n \cap S_1 = \emptyset \quad .$$

<u>Proof</u>: This should be clear from (5.4) and (5.2) for <u>even</u> n, and from (6.6) and (6.4) for <u>odd</u> n. Indeed observe that in either case the facets of Π_n, which are nearest the origin 0, are in HPs of facets of the open cross-polytope

$$S_1 = \{\sum_1^n |x_i| < 1\} \quad .$$

In order to show that $\|x\|_\infty$ may be replaced by $\|x\|_p$ in Theorem 1 we have to establish the following:

<u>Lemma</u> 8. 1. <u>That if</u> $1 \leqq p < \infty$, <u>then</u>

(11.7) $$\Pi_n \cap S_p = \emptyset \quad .$$

2. <u>If</u> Π'_n <u>is a</u> K - S <u>polytope in general position with facets different from the facets of</u> SO_n, <u>then</u>

(11.8) $$\Pi'_n \cap S_p \neq \emptyset \quad .$$

Proof: 1. From (11.6) and (11.5), the equation (11.7) follows immediately.

2. From Theorem 1 we know that

$$\Pi_n' \cap C_n \neq \emptyset \quad , \tag{11.9}$$

hence Π_n' intersects C_n. But then (11.9) and (11.4) clearly imply (11.8), and our proof is complete.

References

1. Coxeter, H. S. M., The derivation of Schoenberg's star-polytopes from Schoute's simplex nets, to appear in the Proceedings of the Coxeter Symposium held in May, 1979, in Toronto.

2. Hardy, G. H., J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, 1934.

3. König, D. and A. Szücs, Mouvement d'un point abandonné à l'intérieur d'un cube, Rendiconti del Circ. Mat. di Palermo, 38 (1913), 79-90.

4. Schoenberg, I. J., Extremum problems for the motions of a billiard ball III. The multi-dimensional case of König and Szücs, MRC TSR #1880, September 1978, Madison, Wisconsin. To appear in Studia Mathematica Hungarica.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2071	2. GOVT ACCESSION NO. AD-A086 550	3. RECIPIENT'S CATALOG NUMBER 9 Technical
4. TITLE (and Subtitle) Extremum Problems for the Motions of a Billiard Ball. IV. A Higher-Dimensional Analogue of Kepler's Stella Octangular, 6		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 I. J. Schoenberg		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024, DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis & Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	11	12. REPORT DATE May 1980
		13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 37		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Extremum problems, Billiard ball motions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Kepler's Stella Octangula (shortened to SO) is the union of the surfaces of the two regular tetrahedra $T = ABCD$ and $T' = A'B'C'D'$ inscribed in the cube

$$\gamma_3 : 1 \leqq x_i \leqq 1, \ (i = 1,2,3)$$

of Figure 1. König and Szücs in [3] have observed that T is obtained by reflexions of the plane of the triangle ABC within γ_3, if we think of the six facets of γ_3 as mirrors which reflect all incident planes back in γ_3.

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ABSTRACT (continued)

A polyhedron obtained by reflexions of a plane in the facets of γ_3 is called a K - S polyhedron. Thus T, and also T', are K - S polyhedra, and SO = T $\cup$ T'.

Let L_3' be a plane intersecting γ_3, which is in general position (G.P.), by which we mean that L_3' is not parallel to any of the three coordinate axes. Let Π_3' denote the K - S polyhedron obtained by reflexions of L_3' in the facets of γ_3, so that $\Pi_3' \subset \gamma_3$. We also say that Π_3' is in G.P. if L_3' is in G.P. We observe that SO does not penetrate within the open cube.

$$C_3 : \|x\|_\infty < \frac{1}{3} ,$$

while the 8 vertices of C_3 are all in SO. It is shown that every K - S polyhedron Π_3 in G.P. and having facets different from the 8 facets of SO, must penetrate into the cube C_3.

Using a result from the previous paper [4] we construct an analogue of the SO in $\mathbb{R}^n$, and denote it by SO_n. This analogue is characterized by a property similar to the above, but with respect to the cube

$$C_n : \|x\|_\infty < \frac{1}{n} \text{ in } \mathbb{R}^n .$$